ISSN 2229-5518

A semi-implicit finite-difference approach for two-dimensional coupled Burgers' equations

Mohammad Tamsir, Vineet Kumar Srivastava

Abstract— In this paper, a semi-implicit finite-difference method is used to find the numerical solution of two-dimensional Coupled Burgers' equation. The proposed scheme forms a system of linear algebraic difference equations to be solved at each time-step. The linear system is solved by direct method. Numerical results are compared with those of exact solutions and other available results. The present method performs well. The proposed scheme can be extended for solving non-linear problems arising in mechanics and other areas of engineering and science.

Index Terms —Burgers' equations; finite- difference; semi-implicit scheme; Reynolds number.

1 Introduction

THE two-dimensional Burgers' equation is a mathematical model which is widely used for various physical applications, such as modeling of gas dynamics and traffic flow, shock waves [1], investigating the shallow water waves[2,3], in examining the chemical reaction-diffusion model of Brusselator[4] etc. It is also used for testing several numerical algorithms. The first attempt to solve Burgers' equation analytically was given by Bateman [5], who derived the steady solution for a simple one-dimensional Burgers' equation, which was used by J.M. Burger in [6] to model turbulence. In the past several years, numerical solution to one-dimensional Burgers' equation and system of multidimensional Burgers' equations have attracted a lot of attention from both scientists and engineers and which has resulted in various finite-difference, finite-element and boundary element methods. Since in this paper the focus is numerical solution of the two-dimensional Burgers' equations, a detailed survey of the numerical schemes for solving the one-dimensional Burgers' equation is not necessary. Interested readers can refer to [7-13] for more details.

Consider two-dimensional coupled nonlinear viscous Burgers' equations:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}-\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=0, \tag{1}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}-\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)=0, \tag{2}$$

subject to the initial conditions:

$$u(x,y,0)=\psi_1(x,y);\ (x,y)\in\Omega,$$
$$v(x,y,0)=\psi_2(x,y);\ (x,y)\in\Omega,$$

and boundary conditions:

$$u(x,y,t)=\xi(x,y,t);\ (x,y)\in\partial\Omega,\ t>0,$$
$$v(x,y,t)=\zeta(x,y,t);\ (x,y)\in\partial\Omega,\ t>0,$$

where $\Omega=\{(x,y): a\le x\le b,\ c\le y\le d\}$ and $\partial\Omega$ is its boundary; $u(x,y,t)$ and $v(x,y,t)$ are the velocity components to be determined, ψ_1,ψ_2, ξ and ζ are known functions and Re is the Reynolds number.
The analytic solution of eqns. (1) and (2) was proposed by Fletcher using the Hopf-Cole transformation [14]. The numerical solutions of this system of equations have been solved by many researchers. Jain and Holla [15] developed two algorithms based on cubic spline method. Fletcher [16] has discussed the comparison of a number of different numerical approaches.Wubs and Goede [17] have applied an explicit–implicit method. Goyon [18] used several multilevel schemes with ADI. Recently A. R. Bahadır [19] has applied a fully implicit method. Vineet etl.[20] have used Crank-Nicolson scheme for numerical solutions of two dimensional coupled Burgers' eqations.
The usual implicit schemes are obviously unconditionally stable with higher order truncation error

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$O((\Delta t)^2+(\Delta x)^2+(\Delta y)^2)$. However, they involve solving a nonlinear algebraic system of equations which makes it inefficient in practice. In this paper, to resolve the above issue, the semi-implicit scheme proposed by Ozis [7] is used for solving two-dimensional Burgers' equations which has a truncation error $O((\Delta t)+(\Delta x)^2+(\Delta y)^2)$. Three numerical experiments have been carried out and their results are presented to illustrate the efficiency of the proposed method.

2 The Solution Procedure

The computational domain Ω is discretized with uniform grid. Denote the discrete approximation of $u(x,y,t)$ and $v(x,y,t)$ at the grid point $(i\Delta x, j\Delta y, n\Delta t)$ by $u_{i,j}^n$ and $v_{i,j}^n$ respectively $(i=0,1,2......,n_x;\ j=0,1,2.....,n_y;$ $n=0,1,2......)$, where $\Delta x=1/n_x$ is the grid size in x-direction, $\Delta y=1/n_y$ is the grid size in y-direction, and Δt represents the increment in time.
Semi-implicit finite-difference approximation to (1) and (2) are given by:

$$\frac{u_{i,j}^{n+1}-u_{i,j}^{n}}{\Delta t}+u_{i,j}^{n}\left(\frac{u_{i+1,j}^{n+1}-u_{i-1,j}^{n+1}}{2\Delta x}\right)+v_{i,j}^{n}\left(\frac{u_{i,j+1}^{n+1}-u_{i,j-1}^{n+1}}{2\Delta y}\right)$$
$$-\frac{1}{\mathrm{Re}}\left[\left(\frac{u_{i+1,j}^{n+1}-2u_{i,j}^{n+1}+u_{i-1,j}^{n+1}}{(\Delta x)^2}\right)+\left(\frac{u_{i,j+1}^{n+1}-2u_{i,j}^{n+1}+u_{i,j-1}^{n+1}}{(\Delta y)^2}\right)\right]=0$$
$$\frac{v_{i,j}^{n+1}-v_{i,j}^{n}}{\Delta t}+u_{i,j}^{n}\left(\frac{v_{i+1,j}^{n+1}-v_{i-1,j}^{n+1}}{2\Delta x}\right)+v_{i,j}^{n}\left(\frac{v_{i,j+1}^{n+1}-v_{i,j-1}^{n+1}}{2\Delta y}\right)$$
$$-\frac{1}{\mathrm{Re}}\left[\left(\frac{v_{i+1,j}^{n+1}-2v_{i,j}^{n+1}+v_{i-1,j}^{n+1}}{(\Delta x)^2}\right)+\left(\frac{v_{i,j+1}^{n+1}-2v_{i,j}^{n+1}+v_{i,j-1}^{n+1}}{(\Delta y)^2}\right)\right]=0$$

The above linear system of equations is solved by direct method.

3 Numerical Examples and Discussion

3.1 Problem 1

The exact solutions of Burgers' equations (1) and (2) can be generated by using the Hopf–Cole transformation [3] which is:

$$u(x,y,t)=\frac{3}{4}-\frac{1}{4[1+\exp(\ (-4x+4y-t)\mathrm{Re}/32\ )]},$$
$$v(x,y,t)=\frac{3}{4}+\frac{1}{4[1+\exp(\ (-4x+4y-t)\mathrm{Re}/32\ )]},$$

Here the computational domain is taken as a square domain $\Omega=\{(x,y):0\le x\le 1, 0\le y\le 1\}$. The initial and boundary conditions for $u(x,y,t)$ and $v(x,y,t)$ are taken from the analytical solutions. The numerical computations are performed using uniform grid, with a mesh width $\Delta x=\Delta y=0.05$. From Tables 1-4, it is clear that the results from the present study are in good agreement with the exact solution for different values of Reynolds number. Comparison of numerical and exact solutions for u and v for $\mathrm{Re}=100$ at $t=0.5$ with $\Delta t=0.001$ are shown in Figs. 1-4.

3.2 Problem 2.

Here the computational domain is taken as $\Omega=\{(x,y):0\le x\le 0.5, 0\le y\le 0.5\}$ and Burgers' equations (1) and (2) are taken with the initial conditions:

$$u(x,y,0)=\sin(\pi x)+\cos(\pi y)$$
$$v(x,y,0)=x+y$$

and boundary conditions:

$$\left.\begin{array}{ll} u(0,y,t)=\cos(\pi y), & u(0.5,y,t)=1+\cos(\pi y) \\ v(0,y,t)=y, & v(0.5,y,t)=0.5+y \end{array}\right\} 0\le y\le 0.5,\ t\ge 0,$$

$$\left.\begin{array}{ll} u(x,0,t)=1+\sin(\pi x), & u(x,0.5,t)=\sin(\pi x) \\ v(x,0,t)=x, & v(x,0.5,t)=x+0.5 \end{array}\right\} 0\le x\le 0.5,\ t\ge 0,$$

The numerical computations are performed using 20×20 grids and $\Delta t=0.0001$. The steady state solutions for $\mathrm{Re}=50$ and $\mathrm{Re}=500$ are obtained at $t=0.625$. Perspective views of u and v for $\mathrm{Re}=50$ at $\Delta t=0.0001$ are given in Figs. 5 and 6 respectively. The results given in Tables 5-8 at some typical mesh points (x,y) demonstrate that the proposed scheme achieves similar results given by [15, 19].

3.3 Problem 3.

In this problem the computational domain is $\Omega=\{(x,y):0\le x\le 1, 0\le y\le 1\}$ and Burgers' equations (1) and (2) are taken with the initial conditions:

$$u(x,y,0)=\frac{-4\pi\cos(2\pi x)\sin(\pi y)}{\mathrm{Re}(2+\sin(2\pi x)\sin(\pi y))},\qquad (x,y)\in\Omega$$

$$v(x,y,0)=\frac{-2\pi\sin(2\pi x)\cos(\pi y)}{\mathrm{Re}(2+\sin(2\pi x)\sin(\pi y))},\qquad (x,y)\in\Omega$$

with boundary conditions:

$$u(0,y,t) = -\frac{2\pi e^{-\frac{5\pi^2 t}{\mathrm{Re}}}\sin(\pi y)}{\mathrm{Re}}, \quad t \ge 0;$$

$$u(1,y,t) = -\frac{2\pi e^{-\frac{5\pi^2 t}{\mathrm{Re}}}\sin(\pi y)}{\mathrm{Re}}, \quad t \ge 0$$

$$u(x,0,t) = 0, \quad t \ge 0; \quad u(x,1,t) = 0, \quad t \ge 0$$

$$v(0,y,t) = 0, \quad t \ge 0; \quad v(1,y,t) = 0, \quad t \ge 0$$

$$v(x,0,t) = -\frac{\pi e^{-\frac{5\pi^2 t}{\mathrm{Re}}}\sin(2\pi x)}{\mathrm{Re}}, \quad t \ge 0;$$

$$v(x,1,t) = \frac{\pi e^{-\frac{5\pi^2 t}{\mathrm{Re}}}\sin(2\pi x)}{\mathrm{Re}}, \quad t \ge 0$$

for which the exact solutions are:

$$u(x,y,t) = -\frac{4\pi e^{-\frac{5\pi^2 t}{\mathrm{Re}}}\cos(2\pi x)\sin(\pi y)}{\mathrm{Re}(2 + e^{-\frac{5\pi^2 t}{\mathrm{Re}}}\sin(2\pi x)\sin(\pi y))}$$

$$v(x,y,t) = -\frac{2\pi e^{-\frac{5\pi^2 t}{\mathrm{Re}}}\sin(2\pi x)\cos(\pi y)}{\mathrm{Re}(2 + e^{-\frac{5\pi^2 t}{\mathrm{Re}}}\sin(2\pi x)\sin(\pi y))}$$

The computed solutions for u and v are ploted in Figs 7 and 9 respestively while the analytical solutions for u and v are shown in Figs 8 and 10 at 20×20 grids and at time level $t = 1.0$ with $\Delta t = 0.001$ for $\mathrm{Re} = 1000$. From these figures it is obvious that numerical solutions are in excellent agreement with the corresponding analytical solutions.

Table 1.
The numerical results for u in comparison with the exact solution at $t = 0.01$ and $t = 1$ with $\Delta t = 0.0001$, and $\mathrm{Re} = 10$.

(x, y)	t=0.01		t=1.0	
	Numerical	Exact	Numerical	Exact
(0.1, 0.1)	0.624805	0.624805	0.605626	0.605626
(0.5, 0.1)	0.594202	0.594202	0.576840	0.576840
(0.9, 0.1)	0.567082	0.567082	0.553017	0.553017
(0.3, 0.3)	0.624805	0.624805	0.605627	0.605626
(0.7, 0.3)	0.594202	0.594202	0.576840	0.576840
(0.1, 0.5)	0.655431	0.655431	0.636685	0.636685
(0.5, 0.5)	0.624805	0.624805	0.605628	0.605626
(0.9, 0.5)	0.594202	0.594202	0.576840	0.576840
(0.3, 0.7)	0.655431	0.655431	0.636687	0.636685
(0.7, 0.7)	0.624805	0.624805	0.605629	0.605626
(0.1, 0.9)	0.682611	0.682611	0.666353	0.666353
(0.5, 0.9)	0.655431	0.655431	0.636687	0.636685
(0.9, 0.9)	0.624805	0.624805	0.605628	0.605626

Table 2.
The numerical results for v in comparison with the exact solution at $t = 0.01$ and $t = 1.0$ with $\Delta t = 0.0001$, and $\mathrm{Re} = 10$.

(x, y)	t=0.01		t=1.0	
	Numerical	Exact	Numerical	Exact
(0.1, 0.1)	0.875195	0.875195	0.894374	0.894374
(0.5, 0.1)	0.905798	0.905798	0.923160	0.923160
(0.9, 0.1)	0.932918	0.932918	0.946983	0.946983
(0.3, 0.3)	0.875195	0.875195	0.894373	0.894374
(0.7, 0.3)	0.905798	0.905798	0.923160	0.923160
(0.1, 0.5)	0.844569	0.844569	0.863315	0.863315
(0.5, 0.5)	0.875195	0.875195	0.894372	0.894374
(0.9, 0.5)	0.905798	0.905798	0.923160	0.923160
(0.3, 0.7)	0.844569	0.844569	0.863313	0.863315
(0.7, 0.7)	0.875195	0.875195	0.894371	0.894374
(0.1, 0.9)	0.817389	0.817389	0.833647	0.833647
(0.5, 0.9)	0.844569	0.844569	0.863313	0.863315
(0.9, 0.9)	0.875195	0.875195	0.894372	0.894374

Table 3
The numerical results for u in comparison with the exact solution at $t = 0.01$ and $t = 1.0$ with $\Delta t = 0.0001$, and $\mathrm{Re} = 100$.

(x, y)	t=0.01		t=1.0	
	Numerical	Exact	Numerical	Exact
(0.1, 0.1)	0.623106	0.623047	0.510307	0.510522
(0.5, 0.1)	0.501617	0.501622	0.500072	0.500074
(0.9, 0.1)	0.500011	0.500011	0.500000	0.500000
(0.3, 0.3)	0.623106	0.623040	0.509824	0.510522
(0.7, 0.3)	0.501617	0.501622	0.500067	0.500074
(0.1, 0.5)	0.748272	0.748274	0.716947	0.716759
(0.5, 0.5)	0.623106	0.623047	0.509499	0.510522
(0.9, 0.5)	0.501617	0.501622	0.500063	0.500074
(0.3, 0.7)	0.748272	0.748274	0.717266	0.716759
(0.7, 0.7)	0.623106	0.623047	0.509314	0.510522
(0.1, 0.9)	0.749988	0.749988	0.749738	0.749742
(0.5, 0.9)	0.748272	0.748274	0.717530	0.716759
(0.9, 0.9)	0.623106	0.623047	0.509172	0.510522

Table 4
The numerical results for v in comparison with the exact solution at $t = 0.01$ and $t = 1.0$ with $\Delta t = 0.0001$, and $\mathrm{Re} = 100$.

(x, y)	t=0.01		t=1.0	
	Numerical	Exact	Numerical	Exact
(0.1, 0.1)	0.876894	0.876953	0.989693	0.989478
(0.5, 0.1)	0.998383	0.998378	0.999928	0.999926
(0.9, 0.1)	0.999989	0.999989	1.00000 0	1.000000
(0.3, 0.3)	0.876894	0.876953	0.990176	0.989478
(0.7, 0.3)	0.998383	0.998378	0.999933	0.999926
(0.1, 0.5)	0.751728	0.751726	0.783053	0.783241
(0.5, 0.5)	0.876894	0.876953	0.990501	0.989478
(0.9, 0.5)	0.998383	0.998378	0.999937	0.999926
(0.3, 0.7)	0.751728	0.751726	0.782734	0.783241
(0.7, 0.7)	0.876894	0.876953	0.990686	0.989478
(0.1, 0.9)	0.750012	0.750012	0.750262	0.750258
(0.5, 0.9)	0.751728	0.751726	0.782470	0.783241
(0.9, 0.9)	0.876894	0.876953	0.990828	0.989478

ISSN 2229-5518

Table 5.
Comparison of computed values of u for $\mathrm{Re} = 50$ at $t = 0.625$.

(x, y)	Present work	A.R.Bahadir	Jain and Holla
(0.1, 0.1)	0.97146	0.96688	0.97258
(0.3, 0.1)	1.15280	1.14827	1.16214
(0.2, 0.2)	0.86308	0.85911	0.86281
(0.4, 0.2)	0.97984	0.97637	0.96483
(0.1, 0.3)	0.66316	0.66019	0.66318
(0.3, 0.3)	0.77232	0.76932	0.77030
(0.2, 0.4)	0.58181	0.57966	0.58070
(0.4, 0.4)	0.75860	0.75678	0.74435

Table 6.
Comparison of computed values of v for $\mathrm{Re} = 50$ at $t = 0.625$.

(x, y)	Present work	A.R.Bahadir	Jain and Holla
(0.1, 0.1)	0.09869	0.09824	0.09773
(0.3, 0.1)	0.14158	0.14112	0.14039
(0.2, 0.2)	0.16754	0.16681	0.16660
(0.4, 0.2)	0.17110	0.17065	0.17397
(0.1, 0.3)	0.26378	0.26261	0.26294
(0.3, 0.3)	0.22655	0.22576	0.22463
(0.2, 0.4)	0.32851	0.32745	0.32402
(0.4, 0.4)	0.32501	0.32441	0.31822

Table 7.
Comparison of computed values of u for $\mathrm{Re} = 500$ at $t = 0.625$.

(x, y)	Present work	A.R.Bahadir	Jain and Holla
(0.15, 0.1)	0.96870	0.96650	0.95691
(0.3, 0.1)	1.03200	1.02970	0.95616
(0.1, 0.2)	0.86178	0.84449	0.84257
(0.2, 0.2)	0.87813	0.87631	0.86399
(0.1, 0.3)	0.67920	0.67809	0.67667
(0.3, 0.3)	0.79945	0.79792	0.76876
(0.15, 0.4)	0.66039	0.54601	0.54408
(0.2, 0.4)	0.58958	0.58874	0.58778

Table 8.
Comparison of computed values of v for $\mathrm{Re} = 500$ at $t = 0.625$.

(x, y)	Present work	A.R.Bahadir	Jain and Holla
(0.15, 0.1)	0.09043	0.09020	0.10177
(0.3, 0.1)	0.10728	0.10690	0.13287
(0.1, 0.2)	0.17295	0.17972	0.18503
(0.2, 0.2)	0.16816	0.16777	0.18169
(0.1, 0.3)	0.26268	0.26222	0.26560
(0.3, 0.3)	0.23550	0.23497	0.25142
(0.15, 0.4)	0.29022	0.31753	0.32084
(0.2, 0.4)	0.30418	0.30371	0.30927

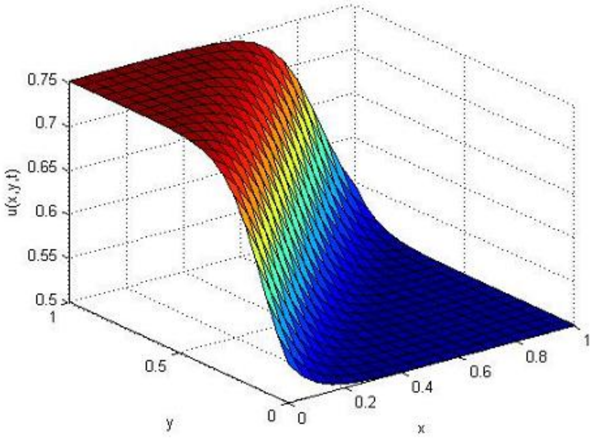

Fig.1. The numerical value of u for $\mathrm{Re} = 100$ at time level $t = 0.5$ with $\Delta t = 0.0001$.

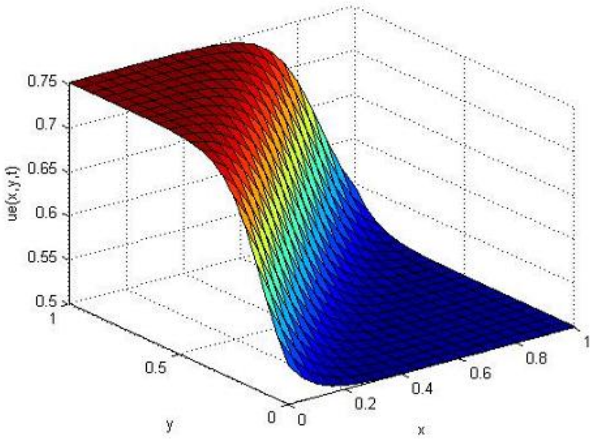

Fig.2. The exact value of u for $\mathrm{Re} = 100$ at time level $t = 0.5$ with $\Delta t = 0.0001$.

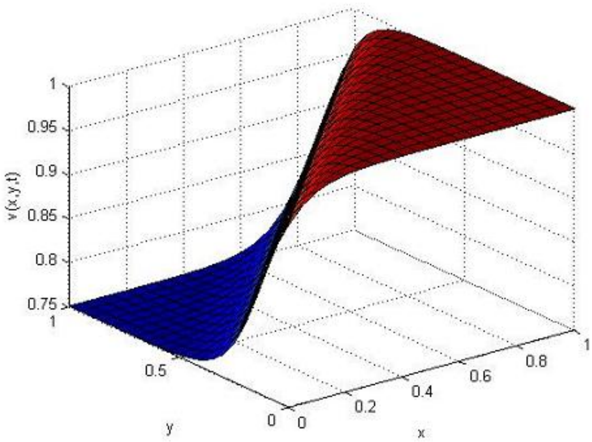

Fig.3. The numerical value of v for $\mathrm{Re} = 100$ at time level $t = 0.5$ with $\Delta t = 0.0001$.

ISSN 2229-5518

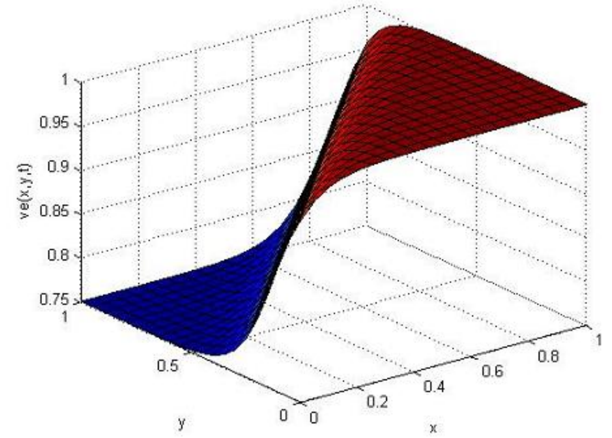

Fig.4. The exact value of v for $\mathrm{Re}=100$ at time level $t=0.5$ with $\Delta t=0.0001$.

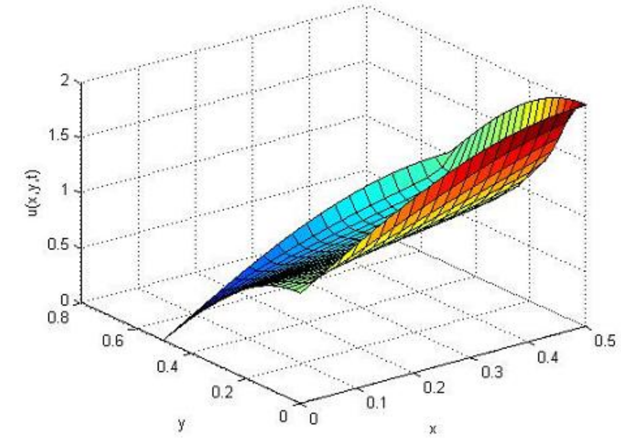

Fig.5. The computed value of u for $\mathrm{Re}=50$ at time level $t=0.625$.

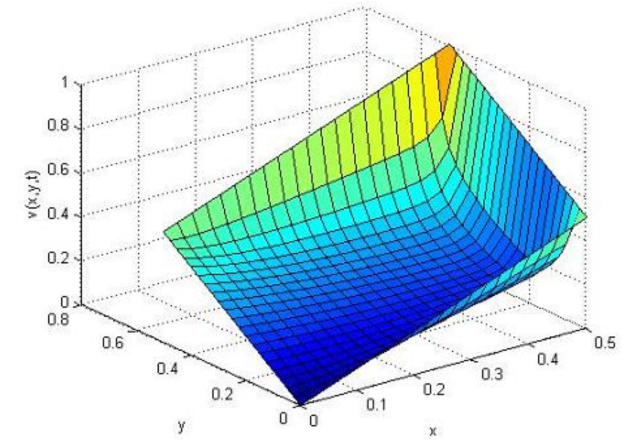

Fig.6. The computed value of v for $\mathrm{Re}=50$ at time level $t=0.625$.

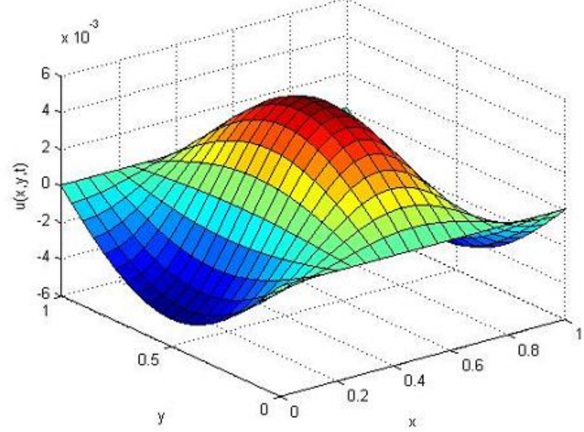

Fig.7. The numerical value of u at 20×20 grids for $\mathrm{Re}=1000$ and at time level $t=1.0$ with $\Delta t=0.001$.

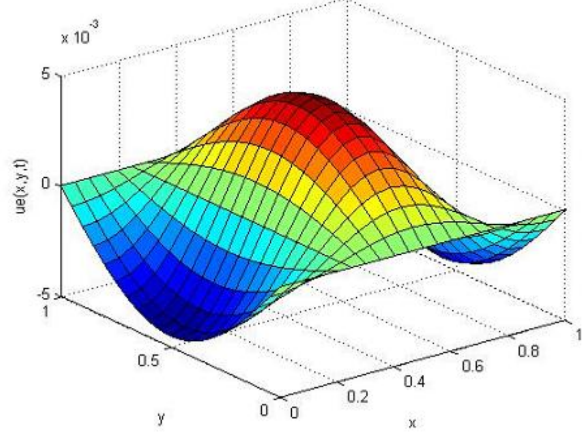

Fig.8. The exact value of u at 20×20 grids for $\mathrm{Re}=1000$ and at time level $t=1.0$ with $\Delta t=0.001$.

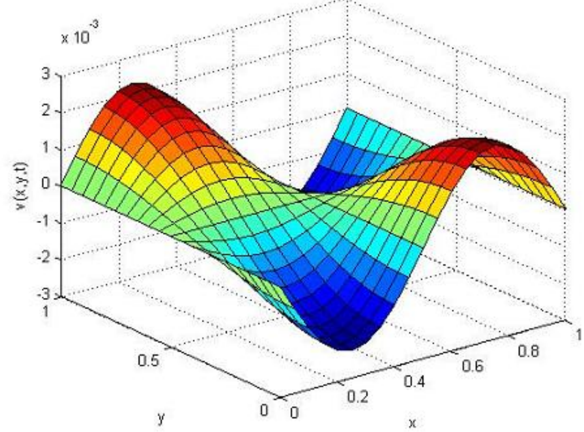

Fig.9. The numerical value of v at 20×20 grids for $\mathrm{Re}=1000$ and at time level $t=1.0$ with $\Delta t=0.001$.

ISSN 2229-5518

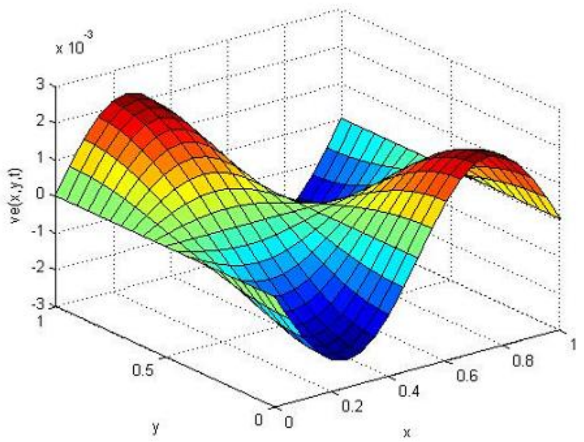

Fig.10. The exact value of v at 20×20 grids for $\text{Re} = 1000$ and at time level $t = 1.0$ with $\Delta t = 0.001$.

4 Conclusion

A semi-implicit finite-difference method based on Ozis [7] has been presented for solving two-dimensional coupled nonlinear viscous Burgers' equations. The efficiency and numerical accuracy of the present scheme are validated through three numerical examples. Numerical results are compared well with those from the exact solutions and previous available results.

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